ON THE STABILITY OF SYSTEMS WITH RANDOM PARAMETERS

(OB USTOICHIVOSTI SISTEM SO SLUCHAINYMI PARAMETRAMI)

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Considered are stability problems for systems containing parameters that are random functions of time. The statement of the problem generalizes a known stability problem of Liapunov [1]. A description is given of the development of the method of Liapunov's functions for the given problem along the lines suggested in [2] on the investigation of the problem of optimal control in systems with random disturbances.*

1. Let us consider the system of differential equations of the perturbed motion

$$dx / dt = f(x, t, y(t)) \tag{1.1}$$

where x is an n-dimensional vector $\{x_1, \ldots, x_n\}$ of generalized coordinates; f is a vector-function $\{f_1, \ldots, f_n\}$; the functions f_i are continuous in all their arguments and satisfy the Lipschitz condition

$$|f_i(x'', t, y(t)) - f_i(x', t, y(t))| \leqslant L \| x'' - x' \| \tag{1.2}$$

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* While this work was in progress it became known through the survey article of Kalman and Bertram [3] that problems on the stability of systems with random parameters were considered in the article of Bertram and Sarachik [4]. This contains definitions of the concept of stability in the mean and theorems which correspond to our results given in Section 4. We call attention to the fact that A.A. Andronov, L.S. Pontriagin, V.V. Stepanov, I.I. Vorovich and a number of other authors have treated stability problems in a manner different from the one used here.

for the variables x_i in the region

$$\|x\| < H, \qquad t \geqslant t_0 \tag{1.3}$$

Here, and in the sequel, $\|x\| = \max(|x_1|, \ldots, |x_n|)$. The function $y(t)$ describes a random Markov process.

In this article we shall confine ourselves to the consideration of the particular case when $y(t)$ is a homogeneous Markov chain with a finite number of states [5, pp. 214-231], i.e. the function $f(y)$ can take on, at every given moment, one value y_i out of a finite set $Y(y_i, \ldots, y_r)$; hereby, the probability $p_{ij}(\Delta t)$ of the change $y_i \to y_j$ during the time Δt satisfies the condition

$$p_{ij}(\Delta t) = \alpha_{ij}\Delta t + o(\Delta t) \qquad (i \neq j) \qquad (\alpha_{ij} = \text{const}) \tag{1.4}$$

where the symbol $o(\Delta t)$ denotes an infinitesimal of higher order than Δt. Without loss of generality, we shall assume that $y_i = i$ $(i = 1, \ldots, r)$. The arguments which follow will remain valid for random functions $y(t)$ of a more general nature, but the derivation of effective criteria of stability becomes quite cumbersome in the general case.

Furthermore, we shall assume that the following equations are satisfied:

$$f_i(0, t, y(t)) \equiv 0 \qquad (y \in Y, \ t \geqslant 0) \tag{1.5}$$

By a solution of the system (1.1) we shall mean an $(n + 1)$-dimensional random vector-function $\{x(x_0, t_0, y_0; t), y(t_0, y_0; t)\}$ whose realizations $\{(x^{(p)}(x_0, t_0, y_0; t), y^{(p)}(t_0, y_0; t)\}$ satisfy Equations (1.1).

The definitions which follow generalize known definitions of stability and of asymptotic stability in the Liapunov sense [1, pp. 19-20]. The generalization is obtained by means of the natural replacement of the usual convergence $x \to 0$, which is the basis of Liapunov's definition, by the convergence in the sense of probability [1, p. 15].

Definition 1.1. The solution $x = 0$ of the system (1.1) (non-perturbed motion) will be said to be stable in the probability sense if for every given $\epsilon > 0$ and $p > 0$ there exists a $\delta > 0$ such that for every solution of the system (1.1) which at the time $t = t_0$ satisfies the inequality

$$\|x_0\| = \|x(t_0)\| \leqslant \delta \tag{1.6}$$

the condition

$$p_t(\|x(x_0, t_0, y_0; t)\| < \varepsilon) > 1 - p \tag{1.7}$$

is satisfied for all $t \geqslant t_0$.

Here, $p_t(\|x\| < \epsilon)$ is the probability that at the time $t \geqslant t_0$ the following inequality is valid*:

$$\| x(x_0, t_0, y_0; t) \| < \varepsilon \quad \text{when } y_0 \in Y$$

The solution $x = 0$ of the system (1.1) is said to be stable in the probability sense on the time interval T for the given estimate $\Delta(\epsilon, p)$ (T is a finite number or $T = \infty$) if the solutions, with the initial conditions satisfying (1.6) for all $t \in [t_0, t_0 + T]$, satisfy the inequality (1.7), and if one can show that $\delta > \Delta(\epsilon, p)$.

Definition 1.2. The unperturbed motion $x = 0$ will be called stable in the probability sense if it is stable by definition (1.1) and if, furthermore, for every given $\eta > 0$ it is true that

$$\lim p_t (\| x \| < \eta) = 1 \quad \text{as} \quad t \to \infty \tag{1.8}$$

for all solutions with the initial conditions

$$\| x_0 \| < H_0 \tag{1.9}$$

where H_0 is some constant which determines the region of attraction of the unperturbed motion.

The following definition of asymptotic stability may also be of interest. In it Equations (1.1) need to be defined only in the finite region (1.3).

Suppose it is known that any arbitrary solution $x(x_0, t_0, y_0; t)$ with $\| x \| \leqslant H_0$, $y_0 \in Y$, $t_0 > 0$, satisfies the condition

$$p_t (\| x(x_0, t_0, y_0; t) \| < H) > 1 - p(H) \tag{1.10}$$

Then we shall say that the solution $x = 0$ of the system (1.1) is $p(H)$-asymptotically stable relative to initial disturbances from the region (1.9) if, in addition to the conditions of definition (1.1), the following conditions are satisfied:

$$\lim p_t (\| x \| < \eta) \geqslant 1 - p(H) \quad \text{as} \quad t \to \infty \tag{1.11}$$

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* Since Equations (1.1) are defined only in a region H which, in general, does not coincide with the entire space, one may assume that the realizations, for which at some time $t = t_1^{(p)}$ the condition $\| x^{(p)} (t_1^{(p)}) \| = \epsilon$ is satisfied, are considered only on the time interval $t_0 \leqslant t < t_1^{(p)}$ when $\| x(t) \| < \epsilon$. Then, the expression $p_t(\| x \| < \epsilon)$ can be considered as the probability of the existence of the realization at the time t.

The unperturbed motion $x = 0$ will be said to be asymptotically stable in the probability sense on the time interval T for the given estimates $p(H)$, $H_0(H)$ and $\tau(\eta, p)$ if every solution $x(x_0, t_0, y_0; t)$ with $\|x\| \leqslant H_0$, $y_0 \in Y$ satisfies the conditions

$$p_t(\|x\| < H) > 1 - p(H) \quad \text{when } t \in [t_0, t_0 + T] \tag{1.12}$$

$$p_t(\|x\| < \eta) \geqslant 1 - p \quad \text{when } t \in [t_0, + \tau(\eta, p), t_0 + T] \tag{1.13}$$

Here it is natural to assume that

$$p \geqslant p(H) \qquad (0 \leqslant \tau \leqslant T) \tag{1.14}$$

From here on, the time instant t_0 will be fixed, unless otherwise specified.

2. Since our aim is to formulate criteria analogous to the theorems of the second method of Liapunov, let us introduce the definitions which for the given case will correspond to the concepts occurring in the use of that method. We shall consider the scalar functions $v(x, t, y)$ which are defined and continuously differentiable in the region (1.3), and which vanish when $x = 0$.

Definition 2.1. The function $v(x, t, y)$ will be said to be positive-definite (negative-definite) if $v(x, t, y) \geqslant w(x)$, $(v(x, t, y) \leqslant - w(x))$ for all $y \in Y$, $t \geqslant t_0$, where $w(x)$ is a positive-definite function in the sense of Liapunov [1, p. 80]. The function $v(x, t, y)$ will be said to be of constant sign if it cannot take on a given sign in the region (1.3). In particular, a function $v(x, t, y)$ which is positive-definite according to Liapunov for any $y \in Y$, where Y is a finite set, will obviously be positive-definite in our sense. If one admits an infinite set of values for y, then in order to have positive-definiteness according to Definition 2.1 it is sufficient to require that the positive-definiteness in the Liapunov sense be uniform in $y \in Y$.

Definition 2.2. The function $v(x, t, y)$ is said to admit an infinitesimally small upper limit if there exists a continuous function $W(x)$ satisfying the conditions

$$v(x, t, y) \leqslant W(x), \quad W(0) = 0 \text{ when } \|x\| < H, \ t \geqslant t_0, \ y \in Y$$

Definition 2.3. The function $v(x, t, y)$ is said to admit an infinitely large lower limit (see [8], p. 36) in the region $\|x\| < H$, if the function $w(x)$ of Definition (2.1) satisfies the condition

$$\lim w(x) = \infty \text{ when } \|x\| \to H$$

By the symbol $M[\psi(a_1, \ldots, a_n);\ a_1, \ldots, a_n/\beta]$ we shall denote the mathematical expectation of the function $\psi(a_1, \ldots, a_n)$ of the random quantities $a_1, \ldots, a_n$ under the condition β, where β denotes some system of equations, inequalities or some other conditions.

Let us consider the solution $\{x(t),\ y(t)\}$ generated by the initial conditions $x = \xi$, $y = \eta$ when $t = \tau$. In accordance with the introduced notation, the expression

$$\mathrm{M}\,[v\,(x\,(t),\ t,\ y\,(t));\ x\,(t)\,y\,(t)\,/\,x\,(t) = \xi,\ y\,(t) = \eta]$$

represents the mathematical expectation of the random function

$$v\,(x\,(\xi,\ \tau,\ \eta;\ t),\ t,\ y\,(\eta,\ \tau;\ t)) \text{ when } t \geqslant \tau$$

Definition 2.4. By the derivative $dM[v]/dt$ of the function v at the point $x = \xi$, $y = \eta$, $t = \tau$ we shall mean, on the basis of Equations (1.1), the limit

$$\frac{d\mathrm{M}\,[v]}{dt} = \lim_{t\to\tau+0} \frac{1}{t-\tau}\,\{\mathrm{M}\,[v\,(x\,(t),\,t,y\,(t));\,x\,(t),y\,(t)\,/\,x\,(\tau) = \xi,\,y\,(\tau) = \eta] - v\,(\xi,\tau,\eta)\} \tag{2.1}$$

In particular, if $F(x,\ y,\ t/\xi,\ \eta,\ \tau)$ is the conditional distribution function for the solution $\{x(t),\ y(t)\}$ [6, p. 283], then

$$\frac{d\mathrm{M}\,[v]}{dt} = \lim_{t\to\tau+0} \frac{1}{t-\tau}\left\{\int_{-\infty}^{+\infty} v\,(x,t,y)\,d_{xy}F\,(x,\,y,\,t/\xi,\,\eta,\,\tau) - v\,(\xi,\,\tau,\,\eta)\right\} \tag{2.2}$$

where the integral is taken in the sense of Stieltjes and evaluated with respect to all the variables $x_1, \ldots, x_n, y$. Because of conditions (1.1), the derivative $dM[v]/dt$ at the point x, $y = j$, t can be evaluated by the formula

$$\frac{d\mathrm{M}\,[v]}{dt} = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i\,(x,\,t,\,y) + \sum_{k\neq j}^{r} \alpha_{jk}\,[v\,(x,\,t,\,k) - v\,(x,\,t,\,j)] \tag{2.3}$$

Note. Formula (2.3) shows that for the computation of the derivative $dM[v]/dt$, just as in the case of ordinary equations, it is not necessary to integrate Equations (1.1), but it is sufficient to know only the right-hand sides of the equations and the probability characteristics of the random process $y(t)$.

3. We shall present some theorems which give sufficient conditions for stability and asymptotic stability in the probability sense.

Theorem 3.1. If it is possible to find for the differential equations (1.1) a positive-definite function $v(x, t, y)$ whose derivative $dM(v)/dt$ is of negative sign, then the solution $x = 0$ is stable in the probability sense.

Proof. Let the number $\epsilon > 0$ and $p > 0$ be given (obviously, one may assume that $\epsilon < H$). Since the function $v(x, t, y)$ is positive-definite, there exists a positive number ϵ_1 such that $v(x, t, y) > \epsilon_1$ when

$$\| x \| = \varepsilon, \qquad y \in Y, \ t \geqslant t_0$$

Let us construct the random function $V(t)$ which we shall use in the proof of the theorem. We shall assume that the realization of the solutions $\{ x^{(p)}(x_0, t_0, y_0; t_0), y^{(p)}(t_0, y_0; t)\}$ generates the realization $V^{(p)}(t)$ of the random function $V(t)$ with the corresponding probability distribution, but only for those values $t > t_0$ for which the following inequality is satisfied:

$$v(x^{(p)}(x_0, t_0, y_0; t), t, y^{(p)}(t_0, y_0; t)) < \varepsilon_1$$

If $t^{(p)}$ is the exact upper boundary of such t, we shall assume that there exists no realization of the solutions $\{ x^{(p)}(t), y^{(p)}(t)\}$ when $t > t^{(p)}$, while the realization $V^{(p)}(t)$ satisfies the condition $V^{(p)}(t) = \epsilon_1$ when $t \geqslant t^{(p)}$.

It is obvious that for the proof of the theorem it is sufficient to show that there exists a number $\delta > 0$ such that if $\| x_0\| < \delta$ then

$$p(V(t) < \varepsilon_1) > 1 - p \tag{3.1}$$

because

$$p(V(t) < \varepsilon_1) \leqslant p_t(\| x(t) \| < \varepsilon) \tag{3.2}$$

Let us determine $\delta > 0$ from the conditions

$$\sup v(x, t_0, y) < p\varepsilon_1 \quad \text{when} \ \| x \| \leqslant \delta \tag{3.3}$$

We shall show that the found number $\delta > 0$ satisfies the conditions of Definition (1.1). Let $\{ x(t), y(t)\}$ be a solution of Equations (1.1) generated by the initial conditions $\{ x_0, y_0 \}$. In accordance with our earlier stipulations we shall suppose that the realizations $\{ x^{(p)}(t), y^{(p)}(t)\}$ of this solution are defined only for those values of t for which they remain within the region $v(x, t, y) < \epsilon_1$.

Let us compute the mathematical expectation $v_t = M[V(t)]$ of the random function $V(t)$. By the definition of v_t we have

$$v_t = \mathrm{M}\,[v(x(t), t, y(t));\ x(t), y(t) \,/\, x(t_0) = x_0,\ y(t_0) = y_0] + \varepsilon_1 p_1(t) \tag{3.4}$$

where $p_1(t)$ is the probability of the break in the realization for $t_0 \leqslant \tau \leqslant t$.

By the hypotheses of the theorem, $v_t \geqslant 0$. Furthermore, we have

$$v_{t+\Delta t} = \mathrm{M}\,[v\,(x\,(t+\Delta t),\, t+\Delta t,\, y\,(t+\Delta t));\, x\,(t+\Delta t),\; y\,(t+\Delta t)\,/\,x\,(t_0)=x_0\; y\,(t_0)=y_0] + p_1\,(t+\Delta t)\,\varepsilon_1 \qquad (3.5)$$

Making use of the property of the process without after-effect [5, p. 86], one may write the next equation (taking into account the break of the realization $\{x^{(p)}(t),\; y^{(p)}(t)\}$ on the surface $v = \epsilon_1$)

$$v_{t+\Delta t} = \mathrm{M}\,[\mathrm{M}\,[v\,(x\,(t+\Delta t),\; t+\Delta t,\; y\,(t+\Delta t));\; x\,(t+\Delta t),\; y\,(t+\Delta t)\,/\,x^\circ(t),\; y^\circ(t)];\; x^\circ(t),\; y^\circ(t)\,/\,x_0,\; y_0] + p_1\,(t+\Delta t)\,\varepsilon_1.$$

The symbols $x^\circ(t)$, $y^\circ(t)$ shall stand for fixed values of $x(t)$, $y(t)$.

Along with the $v_{t+\Delta t}$, we consider the auxiliary quantity $u_{t+\Delta t}$ which we define by the equation

$$u_{t+\Delta t} = \mathrm{M}\,[\mathrm{M}\,[v\,(x^*(t+\Delta t),\; t+\Delta t,\; y\,(t+\Delta t));\; x^*(t+\Delta t),\; y\,(t+\Delta t)\,/\,x^\circ(t),\; y^\circ(t)];\; x^\circ(t),\; y^\circ(t)\,/\,x_0,\; y_0] + p_1\,(t)\,\varepsilon_1$$

where the symbols $x^*(t)$ denote the realizations of solutions without breaks on the surfaces $v = \epsilon_1$.

One can verify that $u_{t+\Delta t} - v_{t+\Delta t} \geqslant o(\Delta t)$. Hence

$$\frac{v_{t+\Delta t} - v_t}{\Delta t} \leqslant \frac{1}{\Delta t}\,\mathrm{M}\,[\{\mathrm{M}\,[v\,(x^*(t+\Delta t),\, t+\Delta t,\; y\,(t+\Delta t));\; x^*(t+\Delta t),\; y\,(t+\Delta t)\,/\,x\,(t),\; y\,(t)] - v\,(x\,(t),\, t,\, y\,(t))\};\; x\,(t),\, y\,(t)\,/\,x_0,\, y_0] + \left|\frac{o\,(\Delta t)}{\Delta t}\right|$$

Taking the limit*

$$\left(\frac{dv_t}{dt}\right)_{dt=+0} \leqslant M\left[\frac{dM_b[v]}{dt}\;;\; x\,(t),\;\; y\,(t)\,/\,x_0,\, y_0\right] \leqslant 0 \qquad (3.6)$$

Since the function v_t is continuous, the inequality (3.4) may be integrated. Hence, we have the inequality

$$v_t \leqslant v\,(x_0,\; t_0,\; y_0) \qquad (3.7)$$

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* The existence of the transformation (3.6) and, in particular, the interchange of the order of taking the limit and of evaluating the mathematical expectation require justification. We note that in the case under consideration this justification does not cause difficulties because of the uniform convergence of the right-hand side of (1.1) to a limit as $t \to \tau + 0$.

Next, let us assume that the theorem is false. Hence, one can find an instant of time $t = T_0 \geqslant t_0$ such that $p(V(t) < \epsilon_1) \leqslant 1 - p$. This means that up to the time $t = T$ the probability of the break-off of the realization $\{x^{(p)}(t), y^{(p)}(t)\}$ is greater or equal to p. But in such a case it is obvious that the next inequality must hold:

$$v_T \geqslant p\varepsilon_1 \tag{3.8}$$

The inequalities (3.3), (3.7) and (3.8) contradict each other. This contradiction establishes the theorem.

Note. Condition (3.3) makes it possible to find a $\delta > 0$ for given ϵ and p by a procedure used in Liapunov's method and described by Chetaev [7, p. 95].

Suppose, for example, that the functions $v(x, y)$ are quadratic forms:

$$v(x, j) = \sum_{i, k=1}^{n} b_{ik}^{(j)} x_i x_k \quad (j = 1, \ldots, r) \tag{3.9}$$

Then

$$\inf v = \min_{(l,j)} \rho_l(j)\, \varepsilon^2 \text{ when } \|x\| = \varepsilon, \quad \sup v = \max_{(l,j)} \rho_l(j)\, \delta^2 \text{ when } \|x\| \leqslant \delta$$

where $\rho_l(j)$ are the roots of the secular equations $\| b_{ik}^{(j)} - \rho\delta_{ik}\| = 0$. Therefore, it is sufficient to determine δ by the conditions

$$\delta < \varepsilon \sqrt{p\, \rho_{\min} / \rho_{\max}} \tag{3.10}$$

This inequality makes it possible to verify the stability on the time interval T for a given estimate $\Delta(\epsilon, p)$.

Theorem 3.2. If for Equations (1.1) there exists a positive-definite function $v(x, t, y)$ which admits an infinitesimally small upper limit and whose derivative, in view of (1.1), is a negative-definite function in the region (1.3), then for every number $p(H) < 1$ there exists a number H_0 such that the solution $x = 0$ of the system (1.1) is $p(H)$ – asymptotically stable relative to the initial disturbances from the region (1.9).

Proof. Let the number $p(H) < 1$ be given. Under the conditions of Theorem (3.2) we have stability in the probability sense. Hence, for some fixed number $\epsilon > 0$, there exists a number $\delta > 0$ such that the inequality

$$p_t(\|x(t)\| < \varepsilon) > 1 - p(H) \tag{3.11}$$

is valid for solutions with the initial conditions $\|x_0\| < \delta$.

Let us choose the number $H_0 = \delta$ and show that it satisfies the conditions of our theorem. For this purpose we take arbitrary numbers

$\eta > 0$ ($\eta < \epsilon$) and $p_1 > 0$. With these numbers we determine the number η_1 so that the next condition be satisfied:

$$[\sup v(x, t, y) \text{ when } \|x\| \leqslant \eta_1] < \left[\tfrac{1}{2} p_1 \inf v(x, t, y) \text{ when } \|x\| \geqslant \eta,\ v(x, t, y) \leqslant \varepsilon_1\right] \tag{3.12}$$

By repeating, with slight modifications, the argument used in the proof of Theorem 3.1, we can verify that

$$p_\tau(\|x(x(t), t, y(t); \tau)\| < \eta) > 1 - \tfrac{1}{2} p_1 \tag{3.13}$$

when $\tau \geqslant t$ for all solutions with the initial conditions

$$x = x(t), \quad y = y(t) \qquad (\|x(t)\| \leqslant \eta_1)$$

Let us show that one can find a sufficiently large value $t = T$ such that the following condition holds:

$$p_T(\|x\| < \eta_1) > 1 - \tfrac{1}{2} p_1 - p(H) \tag{3.14}$$

Indeed, if for all $t > t_0$ it were true that

$$p_t(\|x\| < \eta_1) \leqslant 1 - {}^1\!/_2 p_1 - p(H),$$

then it would follow that $p_t(\eta_1 \leqslant \|x\|,\ v < \epsilon_1) \geqslant p_1/2$.

But then it is easily seen that for all $t > t_0$

$$\frac{dv_t}{dt} \leqslant -\tfrac{1}{2} p_1 \alpha \qquad \left(-\alpha = \inf \frac{dM[v]}{dt} \text{ when } \eta_1 \leqslant \|x\| \leqslant \varepsilon\right)$$

This, however, is impossible because $v_t \geqslant 0$. Thus, from the inequalities (3.11), (3.12) and (3.14) it follows that for any given number p_1 there exists a $T > t_0$ such that for all $t > T$ the next inequality holds:

$$p_t(\|x\| < \eta) > 1 - p(H) - \tfrac{1}{2} p_1 - \tfrac{1}{2} p_1$$

This proves the theorem.

Note 3.1. If $H = \infty$, and if the function $v(x, t, y)$ is defined in the entire space and admits an infinitely large lower limit (see Definition 2.3), while its derivative is a negative-definite function*, then the solution of the system (1.1) is asymptotically stable in the probability sense under disturbances from any bounded region H_0.

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* Everywhere in the sequel, when $H = \infty$, it is assumed that all (almost all) realizations can be continued when $t \to \infty$. See [9, pp. 16-19].

Note 3.2. From the proof of Theorem 3.2 it can be seen how one can verify asymptotic stability in the probability sense on a given time interval for given estimates.

4. Let us continue the consideration of the case when $H = \infty$. In particular, this can always be assumed to be the case for the linear system

$$dx/dt = A(t, y)x \tag{4.1}$$

where x is an n-dimensional vector and $A(t, y)$ is the matrix of the coefficients $a_{ik}(t, y)$.

In the sequel, the symbol $\|x\|_2$ will stand for the expression $\sqrt{(x_1^2 + \ldots + x_n^2)}$. Everything that has been presented earlier in this work is valid, obviously, for this norm.

Definition 4.1. A solution of the system (1.1) will be said to be stable (in the mean square) [5, p. 16] if for every number $\epsilon > 0$ there exists a number $\delta > 0$, such that every solution of the system (1.1), with initial conditions satisfying the inequality

$$\|x_0\|_2 = \|x(t_0)\|_2 \leqslant \delta \tag{4.2}$$

will satisfy the condition

$$\mathrm{M}\,[\|x(x_0, t_0, y_0; t)\|_2^2;\ x(t)/x_0, y_0] < \varepsilon \tag{4.3}$$

for all $t \geqslant t_0$, $y_0 \in Y$.

Definition 4.2. A solution $x = 0$ of the system (1.1) will be said to be asymptotically stable in the mean if it is stable in the mean, and if, in addition, it is true that for all solutions with the initial conditions $\|x\|_2 \leqslant H_0$ the following relation holds:

$$\lim \mathrm{M}\,[\|x(t)\|_2^2] = 0 \quad \text{when} \quad \to \infty \tag{4.4}$$

We shall say in this case that the region H_0 of the space $\{x_i\}$ lies in the region of attraction of the point $x = 0$.

Note. Note that under the stated assumptions stability (asymptotic stability) in the mean of the system (1.1) is a sufficient condition for stability (asymptotic stability) in the probability sense.

Definition 4.3. A solution $x = 0$ of the system (1.1) will be said to be exponentially stable in the mean if for arbitrary initial conditions from the region (1.3) there exist numbers B and a such that for all $t \geqslant t_0$ the following inequality holds:

$$M[\|x(t)\|_2^2;\ x(t)/x_0,\ y_0] \leqslant B\|x_0\|_2^2 \exp(-\alpha(t-t_0)) \tag{4.5}$$

Theorem 4.1. If, for the system (1.1), there exists a function $v(x,\ t,\ y)$ satisfying the conditions

$$c_1\|x\|_2^2 \leqslant v(x,\ t,\ y) \leqslant c_2\|x\|_2^2, \qquad \frac{dM[v]}{dt} \leqslant -c_3\|x\|_2^2 \tag{4.6}$$

where c_1, c_2, and c_3 are positive constants, then the solution $x = 0$ of Equations (1.1) will be exponentially stable in the mean.

Proof. Let us consider a solution of the system (1.1) with the initial conditions $\{x_0,\ y_0\}$. This solution determines a random function of time

$$V(t) = v(x(x_0,\ t_0,\ y_0;\ t);\ t,\ y(t_0,\ y_0;\ t))$$

For the mathematical expectation v_t of the function, we will have the following inequality:

$$c_1 M[\|x(t)\|_2^2;\ x(t)/x_0,\ y_0] \leqslant v_t \leqslant c_2 M[\|x(t)\|_2^2;\ x(t)/x_0,\ y_0]$$

$$\frac{dv_t}{dt} \leqslant -c_3 M[\|x(t)\|_2^2;\ x(t)/x_0,\ y_0] \tag{4.7}$$

From these conditions we obtain by the usual method (see, for example [8], p. 70) the inequality

$$M[\|x(t)\|_2^2;\ x(t)/x_0,\ y_0] \leqslant \frac{c_2}{c_1}\|x_0\|_2^2 \exp\left(-\frac{c_3}{c_2}(t-t_0)\right) \tag{4.8}$$

which proves the theorem.

We note that in the given case there is obtained stability "in the large", i.e. the region of attraction in the mean is the entire space.

Theorem 4.2. If the solution $x = 0$ of Equations (1.1) is exponentially stable in the mean, then in the region $t > t_0$, $y \in Y$ there exists a function $v(x,\ t,\ y)$ which satisfies conditions (4.6).

Proof. Suppose there exist constants B and α such that for any given values x_0 and $t_0 > 0$ the following condition is satisfied:

$$M[\|x(t)\|_2^2;\ x(t)/x_0,\ y_0] \leqslant B\|x_0\|_2^2 \exp(-\alpha(t-t_0)) \tag{4.9}$$

Let us consider the function $v(x,\ t,\ y)$:

$$v(\xi,\ t,\ \eta) = \int_t^\infty M[\|x(\xi,\ t,\ \eta;\ \tau)\|_2^2;\ x(\tau)/x(t) = \xi,\ y(t) = \eta]\, d\tau \tag{4.10}$$

We shall show that this function satisfies the conditions of the theorem. Indeed, by (4.9) we have

$$v(\xi, t, \eta) \leqslant \int_t^\infty B \|\xi\|_2^2 \exp(-\alpha(\tau - t))\, d\tau = \frac{B}{\alpha} \|\xi\|_2^2 = c_2 \|\xi\|_2^2$$

On the other hand, for every realization of a solution of the system (1.1) we have the estimate [9, p. 23]

$$\|x^{(p)}(x_0, t_0, y_0; t)\|_2^2 \geqslant \|x_0\|_2^2 \exp(-2nL(t - t_0))$$

(L is the Lipschitz constant). From this it follows that

$$\mathrm{M}[\|x(t)\|_2^2;\ x(t) / x_0, y_0] \geqslant \|x_0\|_2^2 \exp(-2nL(t - t_0))$$

But then we notice that

$$v(\xi, t, \eta) = \int_t^\infty \mathrm{M}[\|x(\xi, t, \eta; \tau)\|_2^2;\ x(\tau) / \xi, \eta]\, d\tau \geqslant$$

$$\geqslant \int_t^\infty \|\xi\|_2^2 \exp(-2nL(\tau - t))\, d\tau = \frac{1}{2nL} \|\xi\|_2^2 = c_1 \|\xi\|_2^2$$

Thus, the first of conditions (4.6) is satisfied. Let us evaluate $dM[v]/dt$. By Definition 2.4 we have

$$\frac{d\mathrm{M}[v]}{dt} = \lim_{\Delta t \to +0} \frac{1}{\Delta t} \{\mathrm{M}[v(x(t + \Delta t), t + \Delta t, y(t + \Delta t));\ x(t + \Delta t), y(t + \Delta t) / x(t) = \xi, y(t) = \eta] - v(\xi, t, \eta)\}$$

Substituting for $v(x, t, y)$ its expression from (2.10), we obtain

$$\frac{d\mathrm{M}[v]}{dt} = \lim_{\Delta t \to +0} \frac{1}{\Delta t} \Big\{ M\Big[\int_{t+\Delta t}^\infty \mathrm{M}[\|x(x(t + \Delta t), t + \Delta t, y(t + \Delta t); \tau)\|_2^2;\ x(\tau) / x(t + \Delta t), y(t + \Delta t)]\, d\tau;\ x(t + \Delta t), y(t + \Delta t) / x(t), y(t)\Big] - \int_t^\infty \mathrm{M}[\|x(x(t), t, y(t); \tau)\|_2^2;\ x(\tau) / x(t), y(t)]\, d\tau \Big\}$$

Furthermore

$$\mathrm{M}\Big[\int_{t+\Delta t}^\infty \mathrm{M}[\|x(x(t + \Delta t), t + \Delta t, y(t + \Delta t); \tau)\|_2^2;\ x(\tau) / x(t + \Delta t), y(t + \Delta t)]\, d\tau;\ x(t + \Delta t), y(t + \Delta t) / x(t), y(t)\Big] = \int_{t+\Delta t}^\infty \mathrm{M}[\|x(x(t), t, y(t); \tau)\|_2^2;\ x(\tau) / x(t), y(t)]\, d\tau$$

Thus, we obtain

$$\frac{dM[v]}{dt} = \lim_{\Delta t \to +0} \frac{1}{\Delta t}\left\{ \int_{t+\Delta t}^{\infty} M[\|x(x(t), t, y(t); \tau)\|_2^2; x(\tau) / x(t), y(t)]\, d\tau - \right.$$

$$\left. - \int_{t}^{\infty} M[\|x(x(t), t, y(t); \tau)\|_2^2; x(\tau) / x(t), y(t)]\, d\tau \right\} = -\|x(t)\|_2^2 \qquad (4.11)$$

This establishes the theorem.

For the linear system (4.1) with random variables the next theorem is true. It is an adaptation of a theorem of Malkin [10, p. 313] to the case of stability in the mean.

Theorem 4.3. If the solution $x = 0$ of the system (4.1) is exponentially stable in the mean, then for any given positive-definite form $w(x,t,y)$ of the variables $x_1, \ldots, x_n$ whose coefficients $c_{ik}(t, y)$, for all $y \in Y$, are bounded and continuous functions of time, there exists a positive-definite form $v(x, t, y)$ of the same order which satisfies conditions (4.6), and is such that

$$dM[v] / dt = -w(x, t, y) \qquad (4.12)$$

The proof of this theorem is a repetition of the arguments used in the proof of Theorem 4.2. One needs only to select for the function v the function

$$v(\xi, t, \eta) = \int_{t}^{\infty} M[w(x(\xi, t, \eta; \tau), \tau, y(t, \eta; \tau)), x(\tau), y(\tau) / \xi, \eta]\, d\tau \qquad (4.13)$$

Hereby, one should verify that the function $v(\xi, t, \eta)$ is a form in the variables $x_1, \ldots, x_n$. This verification can be carried out in the same way as was done in the cited theorem of Malkin [10, pp. 313-316], since the random solutions of linear equations possess all the properties which were used in [10].

5. Let us consider a system of equations of the form

$$dx / dt = A(t, y)x + R(x, t, y) \qquad (5.1)$$

where the elements of the matrix $A(t, y)$ are continuous bounded functions of time for every $y \in Y$. Relative to the function $R_i(x, t, y)$, we shall assume that in the region (1.3) the following condition holds for every $y \in Y$:

$$|R_i(x, t, y)| \leqslant \gamma \|x\|_2^2 \qquad (\gamma = \text{const} > 0) \qquad (5.2)$$

Alongside with the system (5.1) we consider the system of the first approximation

$$dx / dt = A(t, y) x \tag{5.3}$$

We then have the next theorem which is analogous to a theorem on stability with respect to the first approximation [10, pp. 365-366].

Theorem 5.1. If the solution $x = 0$ of the system (5.3) is exponentially stable in the mean, then the corresponding solution of the system (5.1) is stable in the probability sense, and, furthermore, for every given $p(H)$ the solution $x = 0$ will be $p(H)$ - asymptotically stable for every choice of the function $R(x, t, y)$ satisfying conditions (5.2) in the region (1.3) provided the constant γ is small enough.

Proof. By the conditions of the theorem, there exists a function $v(x, t, y)$ satisfying, because of (5.3), the condition (4.6).

We shall use the symbols $(dM[v]/dt)_{5.1}$ and $(dM[v]/dt)_{5.3}$ to indicate the derivatives of the function $v(x, t, y)$ in view of the systems (5.1) and (5.3), respectively.

Then, in accordance with Formula (2.3), we shall have at the point x, t, j the following relation:

$$\left(\frac{dM[v]}{dt}\right)_{5.1} = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} [a_{i1}(t, j) x_1 + \cdots + a_{in}(t, j) x_n + R_i(x, t, j)] + \\ + \sum_{k \neq j}^{r} \alpha_{jk} [v(x, t, k) - v(x, t, j)] = \left(\frac{dM[v]}{dt}\right)_{5.3} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} R_i(x, t, j) \tag{5.4}$$

Taking into account the fact that the function $v(x, t, y)$ can be chosen as a form in the variables $x_1, \ldots, x_n$, and making use of conditions (4.6) and (5.2), we obtain

$$\left(\frac{dM[v]}{dt}\right)_{5.1} \leqslant -c_3 \|x\|_2^2 + n\gamma\beta \|x\|_2^2 \qquad \left(\beta = \sup \left|\frac{\partial v}{\partial x_i}\right| \text{when } \|x\|_2 = 1\right) \tag{5.5}$$

If γ is sufficiently small, then we obtain, in view of (5.5), the next condition

$$\left(\frac{dM[v]}{dt}\right)_{5.1} \leqslant -\mu \|x\|_2^2 \quad \text{in the region } \|x\|_2 < H \quad (\mu > 0)$$

Thus, the function $v(x, t, y)$ satisfies for the system (5.2) all the conditions of Theorem 3.2.

Note. If the estimate (5.2) is valid in the region $H = \infty$, then the exponential stability in the mean of the system of the first approximation (5.3) will imply the asymptotic stability in the probability sense for the solution $x = 0$ of the system (5.1).

6. Let us now consider the stationary linear system

$$dx / dt = A(y)x \tag{6.1}$$

Theorem 6.1. If the solution $x = 0$ of the system (6.1) is asymptotically stable in the mean, then for every given positive-definite form $w(x, y)$ there exists one, and only one, form $v(x, y)$ of the same order satisfying the condition

$$d\mathrm{M}[v] / dt = -w(x, y) \tag{6.2}$$

Furthermore, this form will, of necessity, be positive-definite.

Proof. Because of Theorem 4.3, it is sufficient to show that if the solution $x = 0$ of the system (6.1) is asymptotically stable in the mean, then it is exponentially stable in the mean. Let us show this.

In the first place, by the hypotheses of the theorem, the solution $x = 0$ is stable in the mean. Therefore, having been given a number $\epsilon > 0$, let us determine a $\delta > 0$ such that the following condition is satisfied:

$$\mathrm{M}[\|x(t)\|_2{}^2;\ x(t) / x(t_0) = x_0,\ y(t_0) = y_0] < \varepsilon, \quad \text{если } \|x_0\| \leqslant \delta \tag{6.3}$$

This stability will be uniform with respect to $t_0 > 0$ and $y_0 \in Y$, because the system is stationary and the set of values of y is finite. Since the system (6.1) is linear, the asymptotic stability in the mean of the unperturbed motion is uniform with respect to x_0. From this it follows, in view of the linearity of the equations, that one can find a $T > 0$ for which the following condition holds:

$$\mathrm{M}[\|x(x_0, t_0, y_0; t_0 + T)\|_2{}^2;\ x(t_0 + T) / x_0 y_0] \leqslant \frac{1}{2}\|x_0\|_2{}^2 \tag{6.4}$$

for every set of values x_0, y_0, $y_0 \in Y$. The computations yield

$$\begin{aligned} &\mathrm{M}[\|x(x_0, t_0, y_0; t_0 + 2T)\|_2{}^2;\ x(t_0 + 2T) / x_0, y_0] = \\ &= \mathrm{M}[\mathrm{M}[\|x((x_0, t_0, y_0; t_0 + T),\ t_0 + T,\ y(t_0, y_0; t_0 + T);\ t_0 + 2T)\|_2{}^2;\ x(t_0 + 2T) / \\ &/ x(t_0 + T),\ y(t_0 + T)];\ x(t_0 + T),\ y(t_0 + T) / x_0, y_0] \leqslant \\ &\leqslant {}^1/_2\mathrm{M}[\|x(x_0, t_0, y_0; t_0 + T)\|_2{}^2;\ x(t_0 + T) / x_0, y_0] \leqslant {}^1/_4\|x_0\|_2{}^2 \end{aligned} \tag{6.5}$$

Continuing the argument by induction, one can obtain for every given positive integer m the condition

$$\mathrm{M}[\|x(x_0, t_0, y_0; t_0 + mT)\|_2{}^2;\ x(t_0 + mT) / x_0, y_0] \leqslant \frac{1}{2^m}\|x_0\|_2{}^2 \tag{6.6}$$

Suppose $t = t_0 + mT + \tau$, where $\tau < T$. Then, making use of the relation [9, p. 23]

$$\| x^{(p)} (x(t_0 + mT),\ t_0 + mT,\ y(t_0 + mT);\ t \|_2^2 \leqslant x \|^{(p)} (t_0 + mT) \|_2^2 \exp(2nL\tau) \qquad (6.7)$$

we obtain

$$\mathrm{M}\, [\| x(x_0,\ t_0,\ y_0;\ t) \|_2^2;\ x(t) / x_0,\ y_0] \leqslant \frac{1}{2^m} \| x_0 \|_2^2 \exp(2nL\tau) \qquad (6.8)$$

Setting $\alpha = 1/T \ln 2$, $B = 2 \exp(2nLT)$, we find that

$$\mathrm{M}\, [\| x(x_0,\ t_0,\ y_0;\ t) \|_2^2;\ x(t) / x_0,\ y_0] \leqslant B \| x_0 \|_2^2 \exp(-\alpha(t - t_0))$$

This completes the proof.

Theorem (6.1) yields a number of algebraic criteria (depending on the choice of the function $w(x, y)$) for asymptotic stability in the mean for the system (6.1). Indeed, let us take any positive-definite quadratic form $w(x, y)$. Let us number its coefficients $c_1(y), \ldots, c_N(y)$ so that $N = n(n + 1)/2$.

If the solution $x = 0$ of the system (6.1) is asymptotically stable in the mean, then, by Theorem 6.1, there exists a positive-definite quadratic form $v(x, y)$ which satisfies condition (6.2). Let us denote the coefficients of this form by $b_1(y), \ldots, b_N(y)$ $(y = 1, \ldots, r)$.

For the determination of these coefficients we obtain a system of N_r linear nonhomogeneous equations

$$A_{i_1}(j) b_1(j) + \cdots + A_{iN}(j) b_N(j) + \sum_{k \neq j}^{r} \alpha_{jk} b_i(k) = -c_i(j) \quad \begin{pmatrix} i = 1, \ldots, N \\ j = 1, \ldots, r \end{pmatrix} \quad (6.9)$$

The coefficients $A_{i1}(j), \ldots, A_{iN}(j)$ are constants that are linear combinations of the coefficients a_{ik} and of the elements of the matrix $A(j)$.

These equations are obtained by equating the coefficients of similar terms in the equations

$$\sum_{s=1}^{n} (a_{si}(j) x_1 + \cdots + a_{sn}(j) x_n) \frac{\partial v(x, j)}{\partial x_s} + \sum_{k \neq j}^{r} \alpha_{jk} [v(x, k) - v(x, j)] = -w(x, j)$$
$$(j = 1, \ldots, r) \qquad (6.10)$$

Thus, for the asymptotic stability in the mean of the system (6.1) it is necessary and sufficient that the forms $v(x, j)$ $(j = 1, \ldots, r)$, with coefficients determined by Equations (6.9), be positive-definite. Making use of Sylvester's criterion for each of these forms, we obtain N_r

algebraic inequalities which guarantee the asymptotic stability in the mean of the system (6.1).

As an example, let us consider the equation

$$dx / dt = a(y)x$$

Here, $a(y)$ is a random function which can take on two values a_1 and a_2, whereby the probability $p_{ij}(\Delta t)$ of the change of values $a_i \to a_j$ is given by Formula (1.4) ($i, j = 1, 2$). Applying the presented method to the given equation, we obtain the system of inequalities

$$a_1 a_2 - \frac{1}{2}(\alpha_{12} a_2 + \alpha_{21} a_1) > 0, \quad a_1 < \frac{1}{2}(\alpha_{12} + \alpha_{21}), \quad a_2 < \frac{1}{2}(\alpha_{12} + \alpha_{21})$$

These inequalities determine the region of asymptotic stability in the mean.

The following assertions are consequences of Theorems 5.1 and 6.1.

Theorem 6.2. If the solution $x = 0$ of the system (6.1) is asymptotically stable in the mean, then the corresponding solution of the equations

$$dx / dt = A(y)x + R(x, t, y) \tag{6.11}$$

will be $p(H)$ - asymptotically stable if condition (5.2) is satisfied and if the constant γ is small enough.

7. In this section we shall consider the problem of stability under random constantly-acting disturbances. Let the equation of the perturbed motion have the form

$$dx / dt = A(y)x + c\eta(t) \tag{7.1}$$

Here, for fixed $y \in Y$, the matrix A has constant coefficients $a_{ij}(y)$; c is an n-dimensional vector with constant components c_i; $\eta(t)$ is a random function which describes the constantly-acting disturbances. We shall restrict ourselves here to random disturbances $\eta(t)$ of a particular type.

Let us assume that the function $\eta(t)$ has the form [11, p. 133]

$$\eta(t) = \sum_k a_k \delta(t - t_k) \tag{7.2}$$

Here t_k is a random quantity having a Poisson distribution along the t-axis with mean frequency λ, i.e.

$$p_m(T) = \frac{(\lambda T)^m}{m!} \exp(-\lambda T)$$

where $p_m(T)$ is the probability that on the time interval of length $T > 0$ there lie m values t_k; a_k are mutually independent, and also independent of t_k, random variables with the same distribution function $F(a)$, and with $M(a) = 0$. The symbol $\delta(t)$ in (7.2) denotes the δ-function. In other words, the random disturbances under consideration are a random selection of impulses of a random quantity. We shall make use of the notation $v = \lambda M\{a^2\}$.

Definition 7.1. An unperturbed motion will be said to be stable in the probability sense (or in probability) under constantly-acting disturbances by Equations (7.1) if, for every given pair of numbers $\epsilon > 0$ and $p >$, there exist two other numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that for every solution $\{x(t), y(t)\}$ of Equations (7.1) with the initial conditions

$$\|x_0\|_2 < \delta_1, \qquad y_0 \in Y$$

the following condition holds:

$$p_t(\|x(t)\|_2 < \varepsilon) > 1 - p, \qquad \text{if} \quad \nu \leqslant \delta_2$$

This definition corresponds in our case to the concept of stability under constantly-acting disturbances for ordinary differential equations [10, pp. 293-294]. We shall show that here, just as in the case of ordinary differential equations, one can make use of Liapunov's function.

Let us suppose that in the absence of random disturbances the linear system of equations

$$dx / dt = A(y)x \tag{7.3}$$

is asymptotically stable in the mean. Then, according to the results of Section 6, there exists a positive-definite quadratic form

$$v(x, y) = \sum_{i,k=1}^{n} b_{ik}(y) x_i x_k$$

whose derivative, by the system (7.3), satisfies the condition

$$\left(\frac{d\mathrm{M}[v]}{dt}\right)_{7.3} = -\sum_{i=1}^{n} x_i^2$$

Let us evaluate the derivative of this function by the use of the complete system of equations (7.1). We obtain

$$\left(\frac{dM[v]}{dt}\right)_{7.1} = \left(\frac{d\mathrm{M}[v]}{dt}\right)_{7.3} + \lim_{\Delta t \to +0} \frac{1}{\Delta t} \{\mathrm{M}[(v(x^{(1)}(t+\Delta t),\ y(t+\Delta t)) - \\ - v(x^{(3)}(t+\Delta t),\ y(t+\Delta t))); / x^{(1)}(t) = x^{(3)}(t),\ y(t)]\} \tag{7.4}$$

where, for the sake of brevity, the solutions of the systems (7.1) and (7.3) are denoted by $x^{(1)}(t)$, and $x^{(3)}(t)$, respectively.

The deviations $\Delta x_i = x_i^{(1)}(t+\Delta t) - x_i^{(3)}(t+\Delta t)$ are caused by the disturbances $\eta(t)$. We note that in the computation of the second term on the right-hand side of Equation (7.4) one may assume that the quantity $y(t)$, for $t < \tau < t+\Delta t$, is constant, because the probable change of $y(t)$ in this term is an infinitesimal of order higher than Δt. Hence, one may write

$$\left(\frac{dM[v]}{dt}\right)_{7.1} = -\sum_{i=1}^{n} x_i^2 + \lim_{\Delta t \to +0} \frac{1}{\Delta t}\left\{M\left[\sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\Delta x_i\right] + M\left[\sum_{i,j=1}^{n} \frac{\partial^2 v}{\partial x_i \partial x_j}\Delta x_i \Delta x_j\right]\right\}$$

The deviation $\Delta x_i(t+\Delta t)$ should be computed by Cauchy's formula for the solution of a nonhomogeneous linear system. Writing this formula in vector notation, we have

$$\Delta x(t+\Delta t) = \int_t^{t+\Delta t} G(t) G^{-1}(\tau) c \eta(\tau)\, d\tau \qquad (7.5)$$

where $G(t)$ is the fundamental matrix of the solutions of the system (7.3) (for a constant value $y(t) = y$).

Since $M\{\eta\} = 0$, we have for our case

$$M\left[\sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\Delta x_i\right] = 0$$

After substituting $\eta(t)$ from (7.2) into the right-hand side of Equation (7.5) we obtain

$$\Delta x_i(t+\Delta t) = \sum_k a_k c_i + O(\Delta t)$$

Therefore

$$M\left[\frac{\partial^2 v}{\partial x_i \partial x_j}\Delta x_i \Delta x_j\right] = b_{ij}(y) \sum_{m=1}^{\infty} p_m(\Delta t)\, M\left(\sum_{k=1}^{m} a_k^2 c_i c_j\right) + o(\Delta t) =$$

$$= b_{ij}(y)\, M\{a^2\} \sum_{m=1}^{\infty} m \frac{(\lambda \Delta t)^m}{m!} \exp(-\lambda \Delta t) + o(\Delta t) = b_{ij}(y)\, c_i c_j M\{a^2\}\, \lambda \Delta t + o(\Delta t)$$

Hence

$$M\left[\sum_{i,j=1}^{n} \frac{\partial^2 v}{\partial x_i \partial x_j}\Delta x_i \Delta x_j\right] = \sum_{i,j=1}^{n} b_{ij}(y)\, c_i c_j \nu \Delta t + o(\Delta t)$$

Thus*

$$\left(\frac{dM[v]}{dt}\right)_{7.1} = -\sum_{i=1}^{n} x_i^2 + \nu \sum_{i,\, j=1}^{n} b_{ij}(y) c_i c_j \tag{7.6}$$

Next, let the numbers $\epsilon > 0$ and $p > 0$ be given. We choose a number $\epsilon_1 > 0$ such that the set of points $v(x, y) < \epsilon_1$ will lie inside the region $\| x \|_2 < \epsilon$ for all $y \in Y$. For the number ϵ_1 we select a $\delta_1 > 0$ such that the surfaces $v(x, y) = \epsilon_1 p$ lie outside the sphere $\| x \|_2 \leqslant \delta_1$ for all $y \in Y$. Let x_0 be some initial point satisfying the condition $\| x_0 \|_2 < \delta_1$. We shall show that the solution $\{ x(x_0, t_0, y_0; t), y(t_0, y_0; t)\}$ satisfies the condition

$$p_t(\| x(t) \|_2 < \varepsilon) > 1 - p \tag{7.7}$$

provided the quantity ν is small enough.

Let us select for this purpose a quantity $\nu > 0$ so small that the derivative (7.6) will be negative in the region $\delta_1 \leqslant \| x \|_2 \leqslant \epsilon$. Let us construct for our proof a random function $Y(t)$ of time, just as was done in the proof of Theorem 3.1.

One can show that for the mathematical expectation v_t of the random function $V(t)$ the following inequality is valid:

$$v_t \leqslant v(x_0, y_0) < p\varepsilon_1 \quad \text{when} \quad t \geqslant t_0 \tag{7.8}$$

Proof. Let the symbol $p(t)$ stand for the probability that $\| x \|_2 \leqslant \delta_1$. We have

$$v_t < \mathrm{M}[v(x(t), y(t));\ x(t), y(t) / \| x(t) \|_2 > \delta_1] + \\ + \mathrm{M}[v(x(t), y(t));\ x(t), y(t) / \| x(t) \|_2 \leqslant \delta_1] + p_1(t)\,\varepsilon_1 \tag{7.9}$$

where $p_1(t)$ is the probability of the break-off of the realization.

The second term in (7.9) satisfies, obviously, the inequality

$$\mathrm{M}[v(x(t), y(t));\ x(t), y(t) / \| x(t) \|_2 \leqslant \delta_1] < p(t)\, p\varepsilon_1$$

For the proof of the inequality (7.8) it is, therefore, sufficient to verify that the first and third terms of (7.9) are smaller than $(1 - p(t) p \epsilon_1$. Let us show this.

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* We take advantage of this opportunity to point out that in [2, Section 5], in the computation of the derivative $dM[v]/dt$ the second term was omitted. Therefore, the deduction that the superposition of a random noise would not change the law of the optimal control [2, Section 5] is false.

Suppose that at some moment of time t the inequality

$$M[v(x(t),\ y(t));\ x(t),\ y(t)\ /\ \|x(t)\|_2 > \delta_1] + p_1(t)\,\varepsilon_1 \leqslant (1 - p(t))\,p\varepsilon_1$$

is satisfied (this inequality certainly holds when $t = t_0$). Then, if Δt is sufficiently small we have

$$M[v(x(t+\Delta t),\ y(t+\Delta t));\ x(t+\Delta t),\ y(t+\Delta t)\ /\ \|x(t+\Delta t)\|_2 > \delta_1] +$$
$$+ p_1(t+\Delta t)\,\varepsilon_1 \leqslant M[v(x^*(t+\Delta t),\ y(t+\Delta t));\ \ x^*(t+\Delta t),\ \ y(t+\Delta t)\ /\ \|x(t)\|_2 > \delta_1] -$$
$$- \Delta p(t)\,p\varepsilon_1 + |\,o(\Delta t)\,|$$

Here, as in the proof of Theorem 3.1, the symbols x^* denote realizations of solutions of the system (7.1) under the assumption that on the tine interval $t \leqslant \tau \leqslant t + \Delta t$ the above-stated rule on their break-off on the surfaces $v = \epsilon_1$ does not apply.

Because $(dM[v]/dt)_{7.1} < 0$ in the region $\|x(t)\|_2 > \delta_1$, the first term of the right-hand side of the last inequality is smaller than

$$M[v(x(t),\ y(t));\ x(t),\ y(t)\ /\ \|x(t)\|_2 > \delta_1] - \alpha\Delta t$$

and hence, for Δt sufficiently small

$$M[v(x(t+\Delta t),\ y(t+\Delta t));\ x(t+\Delta t),\ y(t+\Delta t)\ /\ \|x(t+\Delta t)\|_2 > \delta_1] +$$
$$p_1(t+\Delta t)\,\varepsilon_1 < (1 - p(t))\,p\varepsilon_1 - \Delta p(t)\,p\varepsilon_1 \leqslant (1 - p(t+\Delta t))\,p\varepsilon_1$$

which establishes the inequality (7.8).

From the inequality (7.8) we conclude that in case of asymptotic stability in the mean of the linear system there is stability in the probability sense under constantly-acting random disturbances.

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